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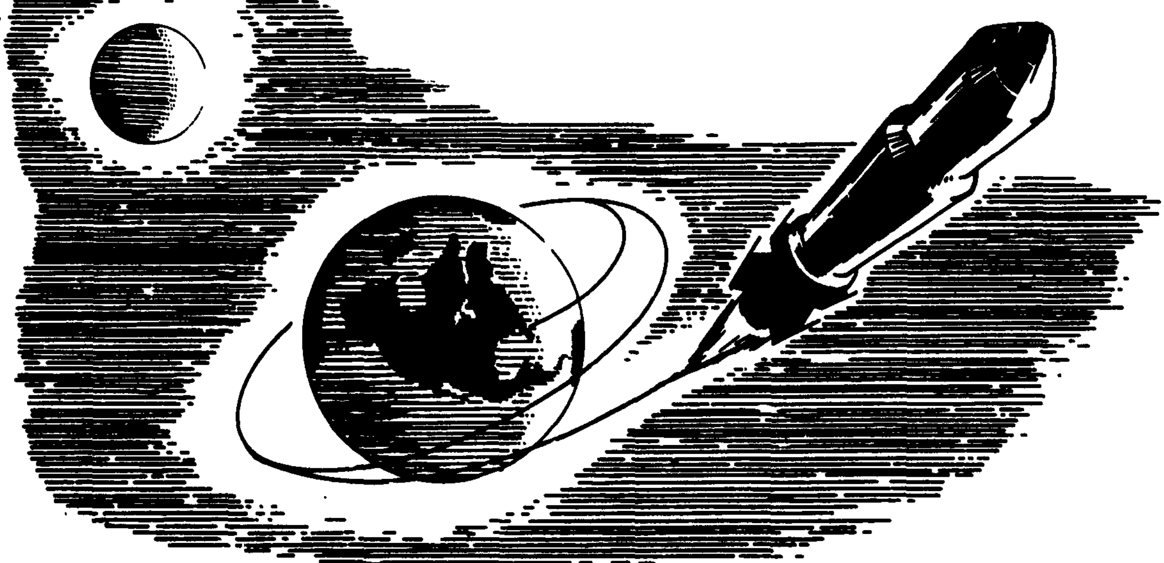
TECHNICAL REPORT

GYROTHEORY OF A SPINNING ROTATIONALLY SYMMETRIC SATELLITE

A New Integral of the Equations of Motion

Herbert Knothe



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GYROTHEORY OF A SPINNING ROTATIONALLY SYMMETRIC SATELLITE

A New Integral of the Equations of Motion

by

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ABSTRACT

Without restriction to plane or infinitesimally small motions, the differential equations for a spinning rotationally symmetric satellite have been established, using new methods, in the form of two second-order differential equations. An integral of these equations has been found.

KEYWORDS

New Integral

Gyrotheory

Spinning Satellite

This report is approved for publication.

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GYROTHEORY OF A SPINNING ROTATIONALLY
SYMMETRIC SATELLITE

A New Integral of the Equations of Motion

Although some of the following deductions can be found dispersed in various American and German textbooks, the whole approach given in this paper, as well as many results, seems to be new. It is worth while to go back in the development to the very beginnings of theoretical physics, i.e., Newton's laws and Hamilton's principle.

We are particularly interested in the motion of the spinning satellite under the influence of the gravitational field of the earth, which shall be assumed as spherically symmetric. Lagrange-Hamilton's theory immediately yields the equations of motion for N mass points:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot{x}_i} = 0$$

$$\frac{\partial L}{\partial y_i} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot{y}_i} = 0 \qquad (1)$$

$$\frac{\partial L}{\partial z_i} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot{z}_i} = 0$$

where

$$L = \sum_{i} \left(\frac{m_i}{2}\,\dot{\bar{x}}_i^{2} - \phi_i \right) + \sum_{i,k=1}^{N} \lambda_{ik} \left(\bar{x}_i - \bar{x}_k \right)^2 \qquad (2)$$

(The vector $\bar{x}_i$ has the components (x_i, y_i, z_i))

and

$$\phi_i = -\frac{kMm_i}{r_i} = -\frac{\mu m_i}{r_i} \qquad (3)$$

means the gravitational potential of the i^{th} mass point. The constants λ_{ik} are Lagrangian multipliers corresponding to the boundary conditions

$$(\bar{x}_i - \bar{x}_k)^2 = \text{const.}$$

which express the fact that for a rigid body the distance between two arbitrary mass points remains constant.

Equations (1) can now be condensed into the vector equation

$$m_i \ddot{\bar{x}}_i + \frac{kMm_i}{{r_i}^3} \bar{x}_i + 2 \sum_{k=1}^{N} \lambda_{ik} (\bar{x}_i - \bar{x}_k) = 0 \qquad (4)$$

We substitute

$$\bar{x}_i = \bar{x}_o + \bar{y}_i \qquad (5)$$

and neglect higher than linear terms of

$$\frac{|\bar{y}_i|}{|\bar{x}_o|} \qquad (6)$$

which is certainly justified because the diameter of the satellite is extremely small compared with the radius of the earth. From equation (5)

it follows

$$r_i^2 = (r_o + \Delta r_i)^2 \approx r_o^2 + 2r_o \, \Delta r_i \approx \bar{x}_o^2 + 2\bar{x}_o \, \bar{y}_i$$

or

$$\Delta r_i = \frac{\bar{x}_i \, \bar{y}_i}{r_o} \tag{7}$$

Therefore, equation (4) can be transformed into

$$m_i \, \ddot{\bar{x}}_o + m_i \, \ddot{\bar{y}}_i = - \frac{\mu m_i}{r_o^3} \bar{x}_o + \frac{3\mu m_i}{r_o^5} (\bar{x}_o \, \bar{y}_i)\bar{x}_o - \frac{\mu m_i}{r_o^3} \bar{y}_i - 2 \sum_{k=1}^{N} \lambda_{ik}(\bar{y}_i - \bar{y}_k) \tag{8}$$

We now sum over all mass points and take into account

$$\sum_{i=1}^{N} m_i \, \bar{y}_i = 0 \tag{9}$$

which states that $\bar{x}_o$ is the center of gravity of the satellite, and

$$\sum_{i,k=1}^{N} \lambda_{ik} \, (\bar{y}_i - \bar{y}_k) = 0 \tag{10}$$

Equation (10) is a consequence of the fact that $\lambda_{ik} = \lambda_{ki}$ since λ_{ik}, λ_{ki} are multipliers belonging to the same boundary condition

$$(\bar{x}_i - \bar{x}_k)^2 = \text{const.}$$

We obtain

$$(\Sigma\, m_i)\ddot{\bar{x}}_o = - \frac{\mu(\Sigma\, m_i)}{r_o^{\,3}}\, \bar{x}_o \tag{11}$$

Equation (11) states that the motion of the center of gravity of the satellite is not influenced by the satellite's librations, at least not as long as we neglect second and higher powers of $|\bar{y}_i| / |\bar{x}_o|$. From equations (8), (9), and (11) we conclude

$$m_i\, \ddot{\bar{y}}_i = + \frac{3\mu\, m_i}{r_o^{\,5}}\, (\bar{x}_o\, \bar{y}_i)\bar{x}_o - \frac{\mu\, m_i}{r_o^{\,3}}\, \bar{y}_i - 2 \sum_{k=1}^{N} \lambda_{ik}\, (\bar{y}_i - \bar{y}_k) \tag{12}$$

If we now form the vector product of (12) with $\bar{y}_i$, sum over i, and take into account that $\lambda_{ik} = \lambda_{ki}$, we obtain

$$\sum m_i\, (\ddot{\bar{y}}_i \times \bar{y}_i) = + \sum \frac{3\mu\, m_i}{r_o^{\,5}}\, (\bar{x}_o\, \bar{y}_i)\, (\bar{x}_o \times \bar{y}_i) \tag{13}$$

or

$$\frac{d}{dt} \sum m_i\, (\dot{\bar{y}}_i \times \bar{y}_i) = + \sum \frac{3\mu\, m_i}{r_o^{\,5}}\, (\bar{x}_o\, \bar{y}_i)\, (\bar{x}_o \times \bar{y}_i) \tag{14}$$

We now make use of a fundamental formula describing the motion of a rigid body:

$$\dot{\bar{y}}_i = \bar{y}_i \times \bar{v} \tag{15}$$

where $\bar{v}$, the instantaneous vector of rotation, is independent of i. Substituting (15) in equation (14) we obtain

$$\frac{d}{dt} \sum m_i \left[(\bar{y}_i^2)\bar{v} - (\bar{v}\,\bar{y}_i)\bar{y}_i \right] = + \sum \frac{3\mu\, m_i}{r_o^5} (\bar{x}_o\, \bar{y}_i)\, (\bar{x}_o \times \bar{y}_i) \qquad (16)$$

For a short time we shall leave the vector representation and shall introduce the principal axes of inertia through the center of gravity as coordinate axes. Because of the rotational symmetry of the satellite, two of these axes are determined only to a rotation about the third axis corresponding to the axis of rotation. They may be chosen arbitrarily but perpendicular to each other and to the third axis.

Let v_1, v_2, v_3 be the components of $\bar{v}$ with respect to such a coordinate system, I_1 being the moment of inertia corresponding to the axis of rotation, $I_2 = I_3$ being the moment of inertia of an axis through the center of gravity and perpendicular to the axis of rotation. Let x_1, x_2, x_3 be the components of $\bar{x}_o$ with respect to our coordinate system. Equation (16) can then be written

$$\begin{aligned} &\frac{d}{dt}(I_1\, v_1) = 0 \\ &\frac{d}{dt}(I_2\, v_2) = -\frac{3\mu}{r_o^5}\, x_1\, x_3\, (I_1 - I_2) \qquad (17) \\ &\frac{d}{dt}(I_2\, v_3) = +\frac{3\mu}{r_o^5}\, x_1\, x_2\, (I_1 - I_2) \end{aligned}$$

We introduce the vectors

$$\bar{\varepsilon} = (v_1, 0, 0)$$

$$\bar{h} = (0, v_2, v_3)$$

and condense equations (17) into the vector equation

$$I_1 \dot{\bar{a}} + I_2 \dot{\bar{h}} = - \frac{3\mu (I_1 - I_2)}{r_o^5 \bar{a}^2} (\bar{x}_o \bar{a}) (\bar{x}_o \times \bar{a}) \tag{18}$$

We still need additional equations in order to eliminate the indeterminacy of $\bar{h}$. At first we have

$$\bar{a} \bar{h} = 0 \tag{19}$$

Secondly we remark that the end point of the unit vector

$$\frac{\bar{a}}{|a|} = \bar{e} \tag{20}$$

whose initial point coincides with the center of gravity, is a point rigidly connected with the gyro satellite. Therefore, formula (15) can be applied to $\bar{e}$:

$$\dot{\bar{e}} = \bar{e} \times \bar{v} = \bar{e} \times (\bar{a} + \bar{h}) = \bar{e} \times \bar{h}$$

or

$$\bar{h} = \dot{\bar{e}} \times \bar{e} = \frac{\dot{\bar{a}} \times \bar{a}}{\bar{a}^2} \tag{21}$$

From equations (19) and (21) we conclude

$$\dot{\bar{a}}\,\bar{h} = \bar{a}\,\dot{\bar{h}} = 0 \tag{22}$$

and forming the scalar product of equation (18) and $\bar{a}$

$$\bar{a}^2 = \text{const.} = \omega_o^{\,2} \tag{23}$$

We replace $\bar{x}_o$ by $r_o\bar{w}$ where $\bar{w}$ is a unit vector indicating the direction earth center → satellite.

Combining equations (18), (20), (21), and (23), we obtain the fundamental differential equation for the motion of the satellite's axis of rotation, represented by the unit vector $\bar{e}$:

$$\boxed{I_1\,\omega_o\dot{\bar{e}} + I_2\,(\ddot{\bar{e}} \times \bar{e}) = \frac{3\mu\,(I_1 - I_2)}{r_o^{\,3}}\,(\bar{w}\,\bar{e})\,(\bar{w} \times \bar{e})} \tag{24}$$

Equation (24) is valid for any elliptical orbit.

After determining $\bar{e}$ from equation (24) $\bar{h}$ can be derived from equation (21) by differentiation and simple algebraical processes. Since

$$\bar{v} = \omega_o\bar{e} + \bar{h}$$

we master the gyro motion of the satellite.

Let us apply formula (24) to the case of a circular orbit where r_o = const. In order to simplify the formulae we first introduce

$$\tau = \Omega_s\,t$$

as a new independent variable where Ω_s means the angular velocity of the satellite's rotation about the earth. Formula (24) assumes the form

$$I_1 \omega_0 \Omega_s \bar{e}' + I_2 \Omega_s^2 (\bar{e}'' \times \bar{e}) = - \frac{3\mu (I_1 - I_2)}{r_0^3} (\bar{w} \bar{e}) (\bar{w} \times \bar{e}) \quad (25)$$

where the primes denote differentiation with respect to τ. Using the abbreviations

$$\begin{aligned} I_1 \omega_0 \Omega_s &= \tilde{I}_1 \\ I_2 \Omega_s^2 &= \tilde{I}_2 \\ - \frac{3\mu}{r_0^3} (I_1 - I_2) &= K \end{aligned} \quad (26)$$

equation (25) can be written

$$\tilde{I}_1 \bar{e}' + \tilde{I}_2 (\bar{e}'' \times \bar{e}) = K (\bar{w} \bar{e}) (\bar{w} \times \bar{e}) \quad (27)$$

It is advantageous to refer $\bar{e}$ to a coordinate system which rotates about the earth in the same way as the satellite's center of gravity. The axes of this coordinate system may be defined by the unit vectors $\bar{w}$, $\bar{w}'$, $\bar{n}$ where $\bar{n}$ is defined by

$$\begin{aligned} \bar{n} &= \bar{w} \times \bar{w}' = \bar{w} \times \bar{t} \\ \bar{t} &= \bar{w}' \end{aligned} \quad (28)$$

The unit vector $\bar{n}$ is a constant vector representing the normal vector

of the orbit plane. We express $\overline{e}$ as a linear combination of $\overline{w}, \overline{t}, \overline{n}$

$$\overline{e} = \lambda\overline{w} + \mu\overline{t} + \rho\overline{n} \tag{29}$$

since

$$\overline{w}' = \overline{t}\ , \qquad \overline{t}' = -\ \overline{w}\ , \qquad \overline{n}' = 0$$

we have

$$\overline{e}' = (\lambda' - \mu)\overline{w} + (\mu' + \lambda)\overline{t} + \rho'\overline{n} \tag{30}$$

If we move with the satellite and observe the attitude of its axis of rotation which is given by the unit vector (λ, μ, ρ) we observe that this vector moves on the surface of the unit sphere at a velocity the vector of which has the components λ', μ', ρ' in the satellite system. Equation (30) shows the relation between the velocity vector $\overline{e}'$ observed in a universe-fixed system and the velocity vector (λ', μ', ρ') observed in a satellite-fixed system can be written

$$\overline{e}' = \overline{z}' + (\overline{n} \times \overline{z}) \tag{31}$$

where

$$\overline{z} = (\lambda, \mu, \rho)$$

and $\lambda^2 + \mu^2 + \rho^2 = 1$.

Applying the operator of equation (31) .

$$\frac{d}{d\tau} + (\overline{n} \times \overline{z})$$

twice, we get

$$\bar{e}'' = \bar{z}'' + 2(\bar{n} \times \bar{z}') - \bar{z} + (\bar{z}\ \bar{n})\bar{n} \tag{32}$$

Equations (31) and (32) enable us to write the equation of motion (27) in the satellite-fixed system $\bar{w}$, $\bar{t}$, $\bar{n}$:

$$\tilde{I}_1 \left(\bar{z}' + (\bar{n} \times \bar{z}) \right) + \tilde{I}_2 \left((\bar{z}'' \times \bar{z}) + 2(\bar{n}\ \bar{z})\bar{z}' + (\bar{z}\ \bar{n})\ (\bar{n} \times \bar{z}) \right) = K\ (\bar{w}\ \bar{z})\ (\bar{w} \times \bar{z}) \tag{33}$$

From equation (33), apparently more complicated than equation (27), an integral of the equation of motion can easily be derived, by forming the vector product of (33) and $\bar{z}'$. We obtain

$$\tilde{I}_1\ (\bar{n}\ \bar{z}') + \tilde{I}_2\ (\bar{z}'\ \bar{z}'') + \tilde{I}_2\ (\bar{n}\ \bar{z})\ (\bar{n}\ \bar{z}') = K\ (\bar{w}\ \bar{z})\ (\bar{w}\ \bar{z}') \tag{34}$$

Integration of equation (34) yields

$$\tilde{I}_1\ (\bar{n}\ \bar{z}) + \frac{1}{2}\ \tilde{I}_2 \left(\bar{z}'^2 + (\bar{n}\ \bar{z})^2 \right) - \frac{1}{2}\ K\ (\bar{w}\ \bar{z})^2 = \text{const.} \tag{35}$$

Let us now write the equation of motion (33) in the coordinate system λ, μ, ρ. The motion can be described by three algebraic differential equations. One of these equations is identical with the integral (35), the second equation is obtained by forming the scalar product of equation (33) and $\bar{n}$, and the third equation expresses the fact that the vector

(λ, μ, ρ) is a unit vector:

$$\tilde{I}_1 \rho + \frac{1}{2} \tilde{I}_2 (\lambda'^2 + \mu'^2 + \rho'^2 + \rho^2) - \frac{1}{2} K\lambda^2 = \text{const.}$$

$$\tilde{I}_1 \rho' + \tilde{I}_2 (\lambda''\mu - \lambda\mu'') + 2\tilde{I}_2 \rho\rho' = K\lambda\mu \tag{36}$$

$$\lambda^2 + \mu^2 + \rho^2 = 1$$

Introducing polar coordinates

$$\lambda = \cos\phi \sin\theta$$

$$\mu = \sin\phi \sin\theta \tag{37}$$

$$\rho = \cos\theta$$

The system (36) is equivalent to

$$\tilde{I}_1 \cos\theta + \frac{1}{2} \tilde{I}_2 (\sin^2\theta\, \phi'^2 + \theta'^2 + \cos^2 \vartheta) - \frac{1}{2} K \sin^2\theta \cos^2\phi = \text{const.}$$

(38)

$$\tilde{I}_1 (\cos\theta)' - \tilde{I}_2 (\sin^2\theta\, \phi')' + \tilde{I}_2 (\cos^2\theta)' = K \sin^2\theta \cos\phi \sin\phi$$

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